Production of Dark Matter in the Early Universe

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Outline

Equilibrium in the Early Universe

- To a good approximation the early Universe was in thermal equilibrium
- Particles have a phase space distribution $f\left(\vec{x}, \vec{p}, t\right)$
- In kinetic and chemical equilibrium f is either a Fermi-Dirac or a Bose-Einstein distribution:

$$f\left(\vec{p}\right) = \left[\exp\left(\left(E-\mu\right) \pm 1\right)\right]^{-1} \quad (1)$$

Freeze Out

- Chemical equilibrium between particle species is governed by annihilation/creation reactions
- Rate depends on cross-section and density
- Rule of thumb for freeze out

$$\begin{aligned} \Gamma &\gtrsim H \quad \Rightarrow \text{thermal equilibrium} \\ \Gamma &\lesssim H \quad \Rightarrow \text{decoupled evolution} \end{aligned}$$

Boltzmann equation

- Evolution of $f(\vec{x}, \vec{p}, t)$ is described by the Boltzmann equation
- For non-interacting, non-relativistic particles:

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \vec{\nabla}_x f + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \vec{\nabla}_p f = 0$$

- For the Friedmann-Robertson-Walker (FRW) model this can be rewritten to

$$\frac{\partial f(E,t)}{\partial t} - \frac{\dot{R}}{R} \frac{|\vec{p}|^2}{E} \frac{\partial f(E,t)}{\partial E} = 0$$

Boltzmann equation (cont.)

- Define number density:

$$n(t) \;=\; \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\; f(E,t)$$

g : internal degrees of freedom

- Integrating the Boltzmann eq. then gives

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3\frac{\dot{R}}{R}n = 0$$

Collision term

- Evolution from equilibrium to decoupled particles depends on interaction
- Introduce collision term:

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3\frac{\dot{R}}{R}n = \frac{g}{(2\pi)^3}\int C[f]\frac{\mathrm{d}^3p}{E}$$

- E.g. for reaction $\psi + a \leftrightarrow i + j$

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} + 3\frac{\dot{R}}{R}n &= -\int \mathrm{d}\Pi_\psi \mathrm{d}\Pi_a \mathrm{d}\Pi_i \mathrm{d}\Pi_j (2\pi)^4 \delta^4(p_\psi + p_a - p_i - p_j)\cdot \\
&\quad \cdot[|M|^2_{\psi+a\to i+j} f_\psi f_a (1\pm f_i)(1\pm f_j) \\
&\quad -|M|^2_{i+j\to\psi+a} f_i f_j (1\pm f_\psi)(1\pm f_a)] \\
\mathrm{d}\Pi &= \frac{g}{(2\pi)^3}\frac{\mathrm{d}^3p}{2E}
\end{aligned}$$

Collision term (cont.)

- Assume stability of the Dark Matter candidate up to the age of the Universe
- Assume CP invariance

$$|M|^2_{i+j\to\psi+a} = |M|^2_{\psi+a\to i+j} := |M|^2$$

- Use Maxwell-Boltzmann statistics

$$\Rightarrow 1 \pm f \simeq 1, \quad f_i(E_i) = \exp\left[-\frac{E_i - \mu_i}{T}\right]$$

$$\begin{aligned}\dot{n}_\psi + 3Hn_\psi &= -\int \mathrm{d}\Pi_\psi \mathrm{d}\Pi_a \mathrm{d}\Pi_i \mathrm{d}\Pi_j (2\pi)^4 \delta^4(p_\psi + p_a - p_i - p_j) \cdot \\ &\quad \cdot |M|^2 [f_\psi f_a - f_i f_j]\end{aligned}$$

Comoving Volume

- Useful to scale out the effect of expansion by using comoving number density $Y = n_\psi / s$ ($sR^3 = \text{const.}$)

$$\Rightarrow s\dot{Y} = \dot{n_\psi} + 3Hn_\psi$$

- Collision term usually depends on temperature $T \rightarrow$ introduce $x = m/T$

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}x} &= -\frac{x}{H(m)s} \int \mathrm{d}\Pi_\psi \mathrm{d}\Pi_a \mathrm{d}\Pi_i \mathrm{d}\Pi_j (2\pi)^4 \delta^4(p_\psi + p_a - p_i - p_j) \cdot \\ & \quad \cdot |M|^2 [f_\psi f_a - f_i f_j] \\ H(m) &= 1.67 g_*^{1/2} \frac{m^2}{m_{\mathrm{Pl}}} \end{aligned}$$

Dark Matter Particle

- More explicitly assume annihilation and inverse annihilation process

$$\begin{aligned} \psi\bar{\psi} &\leftrightarrow X\bar{X} \\ (X &= \text{all possible annihilation products}) \end{aligned}$$

- No asymmetry between ψ and $\bar{\psi}$
- X remain in equilibrium after freeze out of ψ

Equilibrium distribution

- Consider the factor $[f_\psi f_{\bar{\psi}} - f_X f_{\bar{X}}]$
- X, $\bar{X}$ in equilibrium (assuming $\mu = 0$)

$$f_X = \exp(-E_X/T), \qquad f_{\bar{X}} = \exp(-E_{\bar{X}}/T)$$

- The Maxwell-Boltzmann-distribution still valid as the X are in the high energy tail
- Then energy conservation ($E_\psi + E_{\bar{\psi}} - E_X - E_{\bar{X}} = 0$) gives:

$$\begin{aligned} f_X f_{\bar{X}} &= \exp(-(E_X + E_{\bar{X}})/T) \\ &= \exp(-(E_\psi + E_{\bar{\psi}})/T) \\ &= f_\psi^{EQ} f_{\bar{\psi}}^{EQ} \\ \Rightarrow [f_\psi f_{\bar{\psi}} - f_X f_{\bar{X}}] &= [f_\psi f_{\bar{\psi}} - f_\psi^{EQ} f_{\bar{\psi}}^{EQ}] \end{aligned}$$

Reaction cross section

- Now rewrite collision term:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{x\langle\sigma_{\psi\bar{\psi}\to X\bar{X}}|v|\rangle s}{H(m)}\left(Y^2 - Y_{EQ}^2\right)$$

with thermally averaged cross-section times velocity:

$$\begin{aligned}\langle\sigma|v|\rangle &= (n_\psi^{EQ})^{-2}\int \mathrm{d}\Pi_\psi \mathrm{d}\Pi_{\bar{\psi}} \mathrm{d}\Pi_X \mathrm{d}\Pi_{\bar{X}} (2\pi)^4\delta^4 \cdot \\ &\quad \cdot(p_\psi + p_{\bar{\psi}} - p_X - p_{\bar{X}})|M|^2 \exp(-(E_\psi + E_{\bar{\psi}})/T)\end{aligned}$$

Limiting forms

- Look at extreme cases:
- ultra-relativistic

$$Y_{EQ} = \frac{45\zeta(3)}{2\pi^4}\frac{g_{\text{eff}}}{g_{*s}} = 0.278\frac{g_{\text{eff}}}{g_{*s}}$$

- non-relativistic

$$Y_{EQ} = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}\frac{g}{g_{*s}}x^{3/2}e^{-x} = 0.145\frac{g}{g_{*s}}x^{3/2}e^{-x}$$

Final form of the Boltzmann equation

- Note that for radiation dominated Universe $H \propto x^{-2} \quad (\Rightarrow H(T) = x^{-2} H(m))$
- cast Boltzmann eq. in suggestive way:

$$\frac{x}{Y_{EQ}} \frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\Gamma_A}{H} \left[\left(\frac{Y}{Y_{EQ}} \right)^2 - 1 \right]$$

$$\Gamma_A \equiv n_{EQ} \langle \sigma_A |v| \rangle$$

- Note that if $\frac{\Gamma}{H} \lesssim 1 \ \Rightarrow \frac{x \mathrm{d}Y}{Y_{EQ} \mathrm{d}x} \lesssim 1$

$\Rightarrow$ Annihilation freezes out

Final form of the Boltzmann equation (cont.)

- No closed solutions $\rightarrow$ consider qualitative behaviour
- In relativistic regime: $n_{EQ} \propto T^3$, $\Gamma_A \propto T^n$
- In NR regime: $n_{EQ} \propto (mT)^{3/2} \exp(-m/T)$

$\rightarrow$ Γ decreases exponentially $\rightarrow$ freeze out

- But for $\frac{\Gamma}{H} \gtrsim 1 \ \Rightarrow Y \simeq Y_{EQ}$

Cold Dark Matter

- Consider cold dark matter (i.e. DM that was non-relativistic at freeze out)
- For NR species Y_{EQ} decreases exponentially with x
- Parameterize temperature dependence of $\langle \sigma_A |v| \rangle$:

$$
\begin{aligned}
\sigma_A |v| &\propto v^{2n} \quad (n = 0 \text{ for s-wave, } n = 1 \text{ for p-wave, etc.}) \\
\langle v \rangle &\propto T^{1/2} \\
\Rightarrow \langle \sigma_A |v| \rangle &\propto T^n \\
\Rightarrow \langle \sigma_A |v| \rangle &\equiv \sigma_0 \left(\frac{T}{m} \right)^n = \sigma_0 x^{-n} \quad (x \gtrsim 3)
\end{aligned}
$$

Solution of the Boltzmann equation

- The Boltzmann eq. now becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\lambda x^{-n-2}(Y^2 - Y_{EQ}^2) \quad \left(\lambda \equiv \left[\frac{\langle\sigma_A|v|\rangle s}{H(m)}\right]_{x=1}\right)$$

- Define departure from equilibrium Δ

$$\begin{aligned} \Delta &\equiv Y - Y_{EQ} \\ \Rightarrow \Delta' &= -Y'_{EQ} - \lambda x^{-n-2}\Delta(2Y_{EQ} + \Delta) \end{aligned}$$

- At early times both Δ and $|\Delta'|$ are small

$\rightarrow$ Set $\Delta' = 0 \;\Rightarrow \Delta \simeq \frac{x^{n+2}}{2\lambda}$

Solution of the Boltzmann equation (cont.)

- At late times: $\Delta \simeq Y \gg Y_{EQ} \;\rightarrow$ neglect Y'_{EQ} and Y_{EQ}

$$\Rightarrow \Delta' = -\lambda x^{-n-2}\Delta^2$$
$$\Rightarrow Y_\infty = \Delta_\infty = \frac{n+1}{\lambda}x_f^{n+1} = \left[\frac{H(m)(n+1)}{\langle\sigma_A|v|\rangle s}\right]_{x=1} x_f^{n+1}$$

- Need to determine x_f (time of freeze out)
- Define criterion: $\Delta(x_f) = cY_{EQ}$ with c a constant of order unity
- Plug in early time solution $\Delta(x_f) \simeq x_f^{n+2}/\lambda(2+c)$ and solve for x_f

$$x_f \simeq \ln[(2+c)\lambda ac] - \left(n+\frac{1}{2}\right)\ln[\ln[(2+c)\lambda ac]]$$
$$a = 0.145(g/g_{*s})$$

Numerical solution of the Boltzmann equation

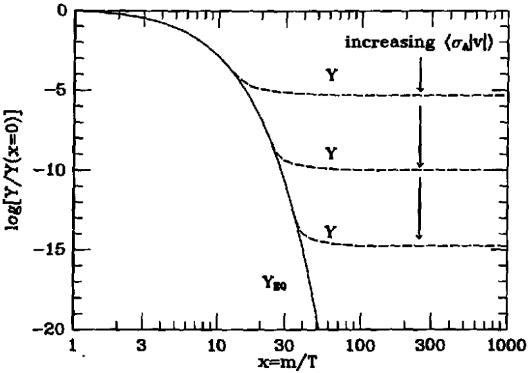

Figure: Freeze out of a massive particle species. Dashed line: actual abundance, solid line: equilibrium abundance [1]

Prediction for Cosmology

- Choosing $c(c+2) = n+1$ gives best fit to numerical results (better than 5%)
- Straightforward calculation then gives:

$$n_{\psi 0} = s_0 Y_\infty = 1.13 \cdot 10^4 \frac{(n+1) x_f^{n+1}}{(g_{*s}/g_*^{1/2}) m_{\mathrm{Pl}} m \sigma_0} \mathrm{cm}^{-3}$$

$$\Omega_\psi h^2 = 1.07 \cdot 10^9 \frac{(n+1)(g_*^{1/2}/g_{*s}) x_f}{m_{\mathrm{Pl}} \langle \sigma_A |v| \rangle}$$

- Note: $\Omega_\psi \propto \langle \sigma_A |v| \rangle^{-1}$

WIMPs as Dark Matter candidates

- Assume ψ has 2 degrees of freedom and $\langle \sigma_A |v| \rangle = a \cdot 10^{-36}\text{cm}^2$ $(n = 0)$ and $g_* = 60$

$$\begin{aligned} \Rightarrow x_f &= 20.3 + \ln\left[a\left(\frac{m}{\text{GeV}}\right)\right] \\ Y_\infty &= \frac{3 \cdot 10^{-10}}{a(m/\text{GeV})} \\ \Omega_\psi h^2 &= \frac{0.043 x_f}{a} \simeq \frac{0.087}{a} \end{aligned}$$

- Hence for Ω_ψ of order unity the annihilation cross section needs to be characteristic of a weak process.

Restriction of SUSY parameters from Cosmology

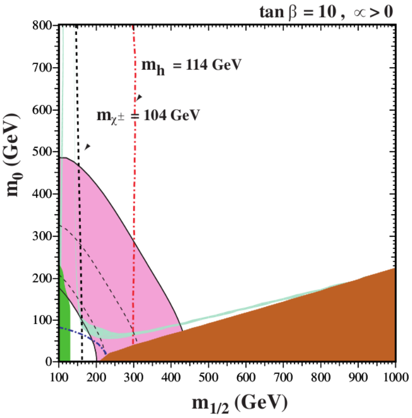

Figure: The $(m_{1/2}, m_0)$ plane for $\tan\beta = 10$. The turquoise shaded area is preferred by cosmology. [2]

Exceptions in the calculation

- There are exception to the simple T-dependence assumed above [3]
  - Neutralino DM with an additional sparticle

  $\rightarrow$ Annihilation of the sparticle can control the relic abundance if $\Delta m \sim T_f$
  - DM particle lies near a mass threshold for annihilation to an additional species

  $\rightarrow$ "Forbidden" annihilation channel can dominate the cross section
  - Annihilation takes place near a pole in the cross section.

  $\rightarrow$ Special care needs to be taken in the thermal averaging of the cross section.

Hot Dark Matter

- Now consider a particle species that decoupled while still relativistic (e.g. neutrinos)
- Y_{EQ} does not change with time

→ Y_∞ insensitive to the details of freeze out

- Y_∞ is just the equilibrium value at decoupling

$$Y_\infty = Y_{EQ}(x_f) = 0.278 \frac{g_{\text{eff}}}{g_{*s}(x_f)}$$

Neutrinos as Dark Matter?

- For a particle species ψ this gives:

$$\Omega_\psi h^2 = 7.83 \cdot 10^{-2} \frac{g_{\text{eff}}}{g_{*s}} \left(\frac{m}{\text{eV}}\right)$$

- Light neutrinos decouple when $T \sim$ few MeV and $g_{*s} = g_* = 10.75$. A two-component neutrino species has $g_{\text{eff}} = 1.5$ and hence

$$\Omega_{\nu\bar{\nu}} h^2 = \frac{m_\nu}{91.5\text{eV}}$$

- For neutrinos to make up all the measured dark matter, the mass needs to be of order 10 eV
- Experiments suggest lower neutrino masses

Conclusion

- Relic particle abundances can be calculated with the Boltzmann equation
- Abundance of cold relics depends (almost) only on annihilation cross section
- WIMPs are good dark matter candidates
- Neutrinos as hot dark matter can be excluded because of low mass
- Caveat: all calculation depend on the underlying model for cross section

References

[1] M.E. Turner and E.W. Kolb, The Early Universe

[2] J. Ellis and K.A. Olive, Supersymmetric Dark Matter Candidates, arXiv:1001.3651v1

[3] K. Griest and D. Seckel, Three exceptions in the calculation of relic abundances, Phys. Rev. D 43, 3191–3203 (1991)